Search-based and Stochastic Solutions to the Zonotope and Ellipsotope Containment Problems

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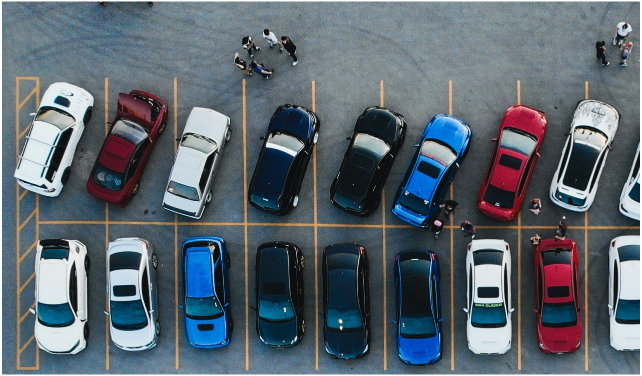
June 26, 2024

A Frustrating Situation...



Source: `https://unsplash.com/photos/cars-parked-on-parking-lot-during-daytime-vMneecAwo34`

Part I: Basics about Zonotopes

Definition

Definition: Zonotopes

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where $\vec{c} \in \mathbb{R}^n$ is the *center* and $\underline{G} \in \mathbb{R}^{n \times m}$ is the matrix of *generators*.

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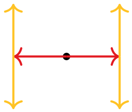

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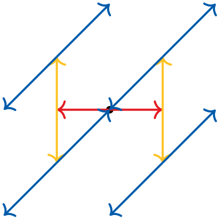

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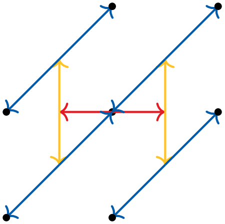

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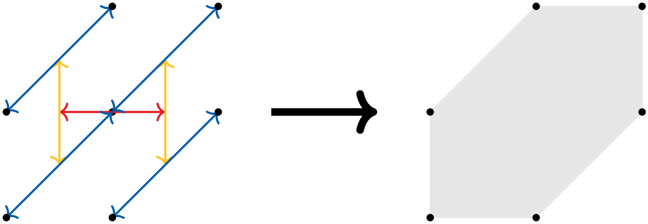
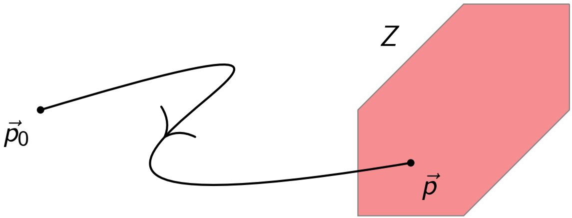


Example: Robust Control



Question: How to check if $\vec{p} \in Z$?

Solving the Point Containment Problem

$$\vec{p} \in Z = \left\{ \underline{G}\vec{\beta} + \vec{c} \,\middle|\, \vec{\beta} \in [-1, 1]^m \right\}$$

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\end{aligned}
$$

However, solving this gives much more information than containment: It measures, *how far away the point is from Z*.

Zonotope Norms

The function

$$\|\vec{p}\|_Z = \min_{\vec{\beta}} \|\vec{\beta}\|_\infty \text{ subject to } \underline{G}\vec{\beta} = \vec{p}$$

is a *norm* on $\mathbb{R}^n$ (and thus *convex*) if $Z = \langle \vec{c}, \underline{G} \rangle$ is non-degenerate.

Zonotope Containment Problem

How to check whether a zonotope $Z_1 = \langle \vec{c}_1, \underline{G}_1 \rangle$ is inside a zonotope $Z_2 = \langle \vec{c}_2, \underline{G}_2 \rangle$?

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Part II: Search-Based Vertex Enumeration

Bauer Maximum Principle

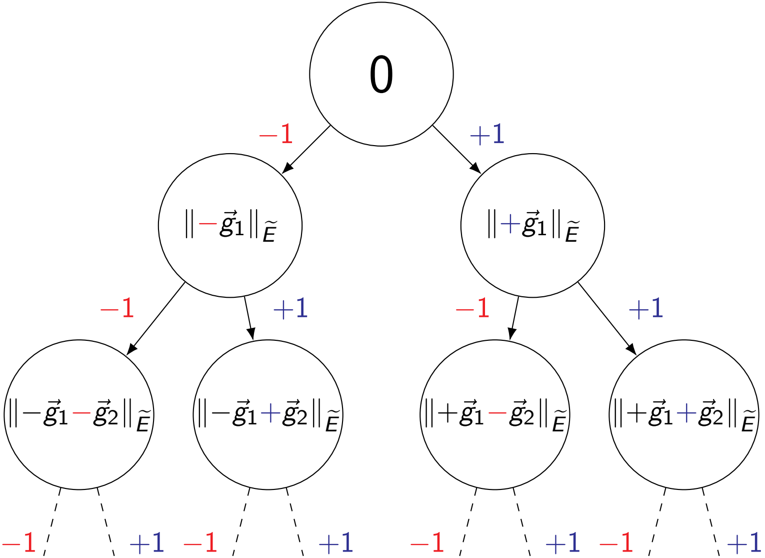
Theorem: Bauer Maximum Principle

A continuous, convex function $f : S \to \mathbb{R}$ on a non-empty, convex, compact set S attains its maximum on an extreme point (e.g., a vertex) of the set S.

$$\max_{\|\vec{\alpha}\|_\infty \leq 1} \min_{\underline{G}_2 \vec{\beta} = \vec{p} - \vec{c}_2} \|\vec{\beta}\|_\infty$$

$\Rightarrow$ the maximum is attained at a point $\vec{\alpha} \in \{-1, +1\}^{m_1}$

Tree Search


0
−1
+1
$\|-\vec{g}_1\|_{\widetilde{E}}$
$\|+\vec{g}_1\|_{\widetilde{E}}$
−1
+1
−1
+1
$\|-\vec{g}_1-\vec{g}_2\|_{\widetilde{E}}$
$\|-\vec{g}_1+\vec{g}_2\|_{\widetilde{E}}$
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−1
+1
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Eliminating Branches

Remember: The cost function is a *norm* $\Rightarrow$ Triangle inequality!

$$\|\vec{a} - \vec{b} \pm \vec{c} \pm \vec{d}\|_{Z_2}$$

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$$\|\vec{a} - \vec{b} \pm \vec{c} \pm \vec{d}\|_{Z_2} \leq \underbrace{\|\vec{a} - \vec{b}\|_{Z_2}}_{\text{Node value}} + \underbrace{\|\vec{c}\|_{Z_2} + \|\vec{d}\|_{Z_2}}_{\text{Worst case additional cost}}$$

Eliminating Branches - Example

Assume $\|\vec{g}_1\|_{Z_2} = 10$, $\|\vec{g}_2\|_{Z_2} = 9$, $\|\vec{g}_3\|_{Z_2} = 2$, $\|\vec{g}_4\|_{Z_2} = 1$

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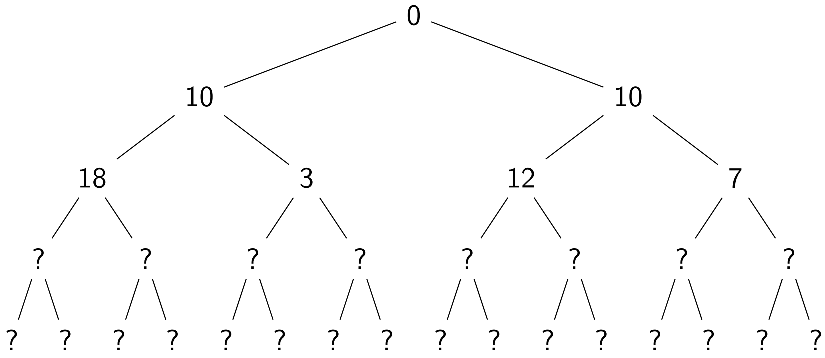

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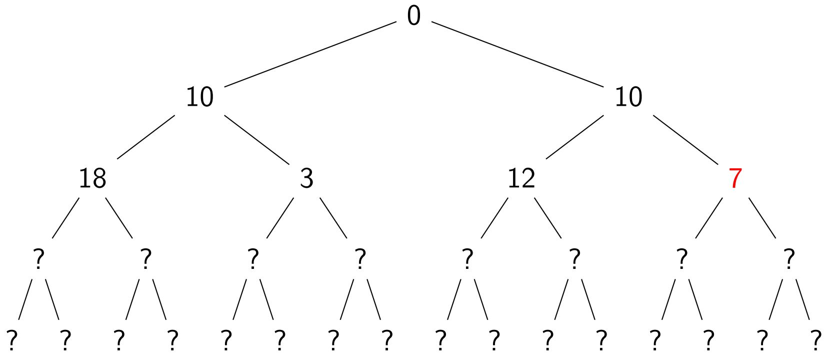

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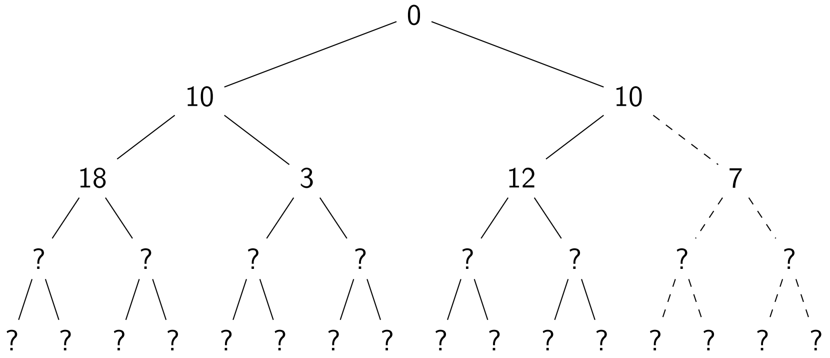

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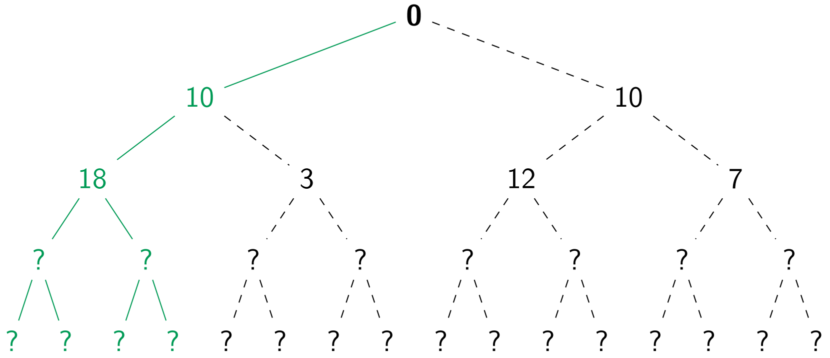
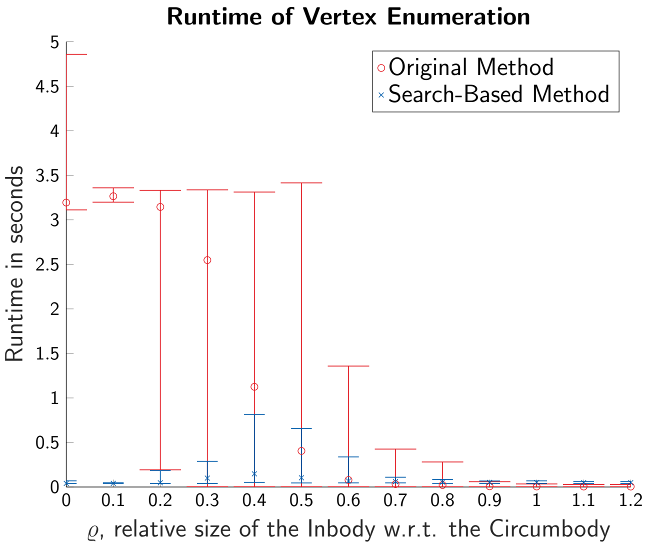


Numerical Results - Tree Search



Ellipsotopes

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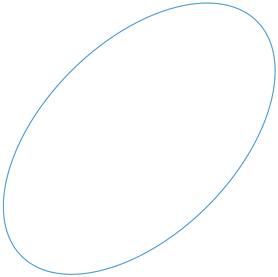
$\Rightarrow$ Tree-Search can be generalized to the case where the outer set is an ellipsotope!

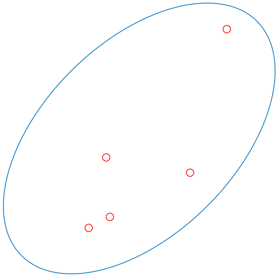
Part III: Halfspace Sampling

Halfspace Sampling



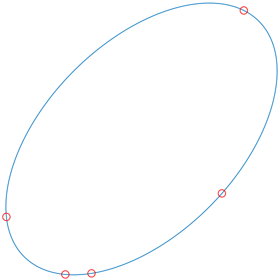
Halfspace Sampling

Step 1: **Uniform** Sampling of the circumbody



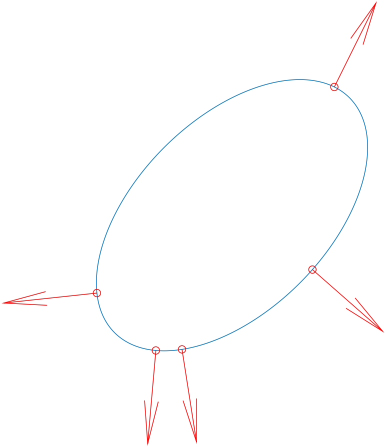
Halfspace Sampling

Step 2: Move points outwards, to the boundary



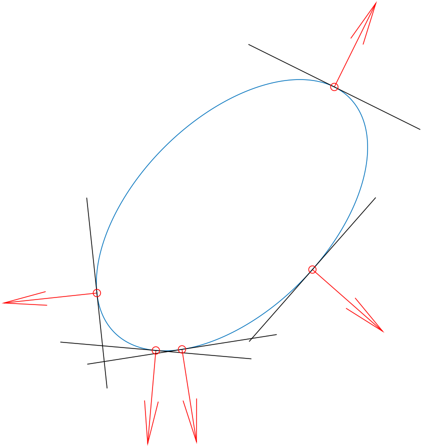
Halfspace Sampling

Step 3: Compute normal vectors



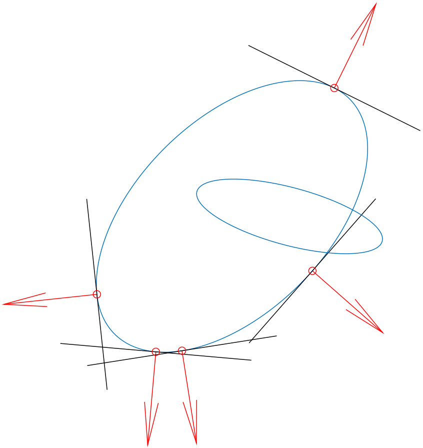
Halfspace Sampling

Step 4: Compute supporting halfspaces



Halfspace Sampling

Step 5: Check for (partial) containment



Shenmaier Sampling - Actual Theorem

Shenmaier Sampling (improved)

Let $K \subset \mathbb{R}^n$ be compact with $\mathrm{vol}(K) > 0$, $\vec{c} \in \mathbb{R}^n$ a vector, $\|\cdot\|$ some norm on $\mathbb{R}^n$, and $\mathcal{B}$ the unit ball of the norm $\|\cdot\|$. For $N \in \mathbb{N}$, let $\vec{\mathfrak{v}}_i \in \mathcal{B}$ be independently, ϑ-uniformly sampled on $\mathcal{B}$, for $i = 1, \ldots, N$, and let

$$\vec{\mathfrak{z}}_i = \operatorname*{argmax}_{\vec{z} \in \mathcal{B}^*} \vec{z}^\top \vec{\mathfrak{v}}_i.$$

We define the approximation

$$\mathfrak{r}_{\mathsf{S}} := \max_i \max_{\vec{x} \in K} \vec{\mathfrak{z}}_i^\top \vec{x}.$$

Then

$$\mathfrak{r}_{\mathsf{S}} \leq \max_{\vec{x} \in K} \|\vec{x}\|$$

always holds, and for any $\varepsilon \in (0, 1]$ we have

$$(1 - \varepsilon) \cdot \max_{\vec{x} \in K} \|\vec{x}\| \leq \mathfrak{r}_{\mathsf{S}}$$

with a probability of at least

$$P_{\mathsf{S}}(\varepsilon) := 1 - \left(1 - \left(\frac{\varepsilon}{2 + \varepsilon}\right)^n + \vartheta\right)^N.$$

Halfspace Sampling - A Probabilistic Method

Halfspace Sampling - Correctness

Suppose the halfspace sampling algorithm has been used in $\mathbb{R}^n$ with N sampled points, and the maximal length in any computed direction was $\mathfrak{r} < 1$. Then containment holds with a probability of at least

$$P \geq 1 - \left(1 - \left(\frac{1-\mathfrak{r}}{3-\mathfrak{r}}\right)^n\right)^N.$$

Halfspace Sampling - Important Details

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- The point distribution must be **uniform** $\rightarrow$ Ball Walk, Hit-and-Run, Billiard Walk, etc, all easy to evaluate on ellipsotopes

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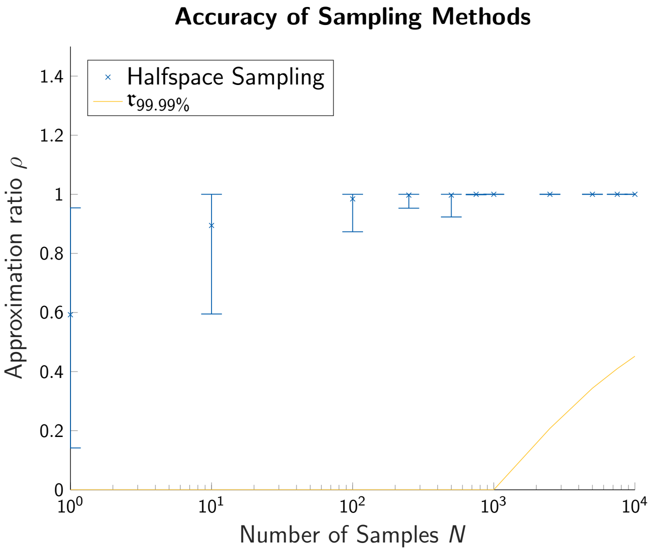
- For zonotopes, reduces to uniform sampling of facets

Numerical Results - Halfspace Sampling



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- Can the bound on the probability be improved?
- Can the probabilistic method be de-randomized?

Runtime Tree Search vs. Halfspace Sampling

Average runtimes of each algorithm, for $N = 10^4$ samples. The average is over 100 zonotope-pairs in dimension 5, with 10 generators. All values are displayed in milliseconds.

Algorithm	Tree Search	Halfspace Sampling
Runtime	466.6±24.5	93.1±0.2