On the co-NP-completeness of the Zonotope Containment Problem

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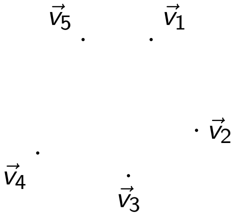
Polytopes

A (non-degenerate) *polytope* P is either given by its vertices



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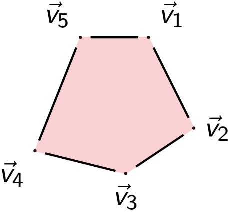

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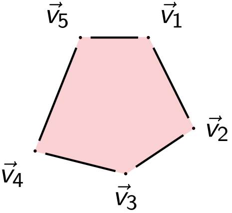

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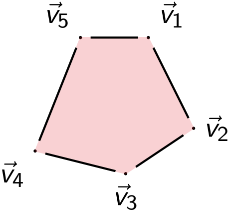

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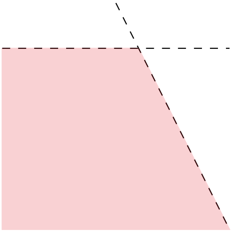

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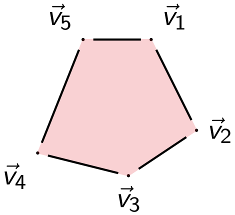

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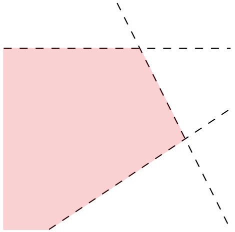

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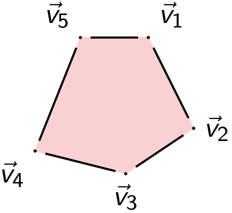

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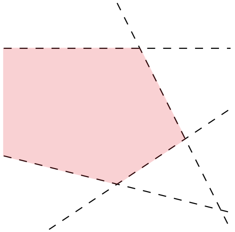

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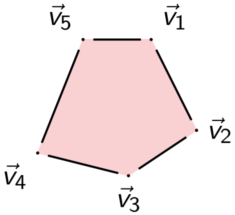

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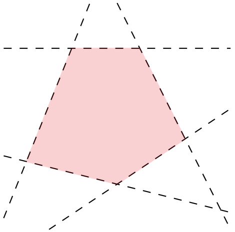

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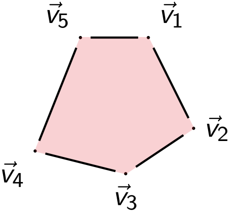

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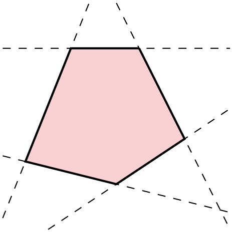

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A *zonotope* $Z = \langle \vec{c}, \underline{G} \rangle$ is a set of the form

$$Z = \left\{ \underline{G}\vec{\beta} + \vec{c} \;\middle|\; \vec{\beta} \in [-1, 1]^m \right\},$$

where $\vec{c} \in \mathbb{R}^n$ is the *center* and $\underline{G} \in \mathbb{R}^{n \times m}$ is the matrix of *generators*. It is *non-degenerate* if $\operatorname{rank}(\underline{G}) = n$, and a *parallelotope* if $m = n$.

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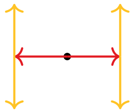

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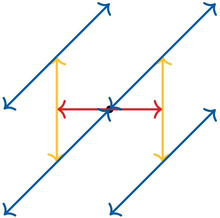

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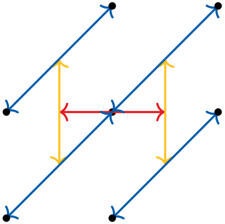

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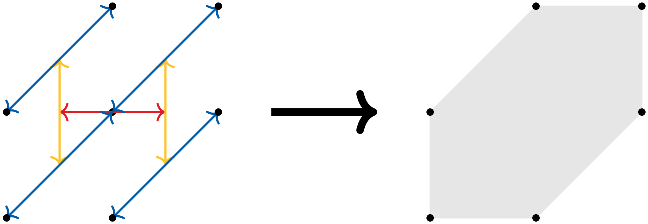
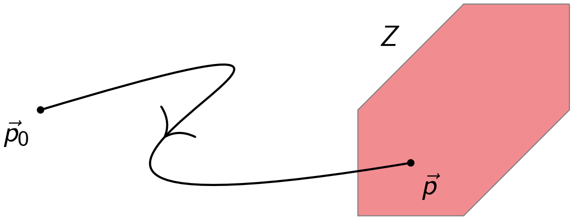


Example: Robust Control



Question: How to check if $\vec{p} \in Z$?

Solving the Point Containment Problem

$$\vec{p} \in Z = \left\{ \underline{G}\vec{\beta} + \vec{c} \,\middle|\, \vec{\beta} \in [-1, 1]^m \right\}$$

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$$\Leftrightarrow \quad \min_{\vec{z}} \begin{bmatrix} 1 & \vec{0}_m^T \end{bmatrix} \vec{z}, \ \text{ s.t. } \begin{cases} \begin{pmatrix} \vec{0}_n & \underline{G} \end{pmatrix} \vec{z} = \vec{p} - \vec{c}, \\ \begin{pmatrix} -\vec{1}_m & \underline{I}_{m\times m} \\ -\vec{1}_m & -\underline{I}_{m\times m} \end{pmatrix} \vec{z} \leq \vec{0}_{2m}. \end{cases}$$

However, solving this gives much more information than containment: It measures, *how far away the point is from Z*.

Zonotope Norms

The function

$$\|\vec{p}\|_Z = \min_{\vec{\beta}} \|\vec{\beta}\|_\infty \text{ subject to } \underline{G}\vec{\beta} = \vec{p}$$

is a *norm* on $\mathbb{R}^n$, if $Z = \langle \vec{c}, \underline{G} \rangle$ is non-degenerate.

The unit ball of $\|\cdot\|_Z$ coincides with Z:

$$Z = B_1^{\|\cdot\|_Z}(\vec{c}) := \{\vec{x} \in \mathbb{R}^n | \|\vec{x} - \vec{c}\|_Z \leq 1\},$$

and similarly for the boundary ∂Z of Z:

$$\partial Z = \partial B_1^{\|\cdot\|_Z}(\vec{c}) = \{\vec{x} \in \mathbb{R}^n | \|\vec{x} - \vec{c}\|_Z = 1\}.$$

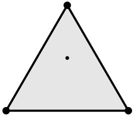
Dual Polytopes

The *dual polytope* P^Δ of a polytope P that contains the origin is the polytope one gets by replacing facets by vertices, and vertices by facets, i.e., if

$$P = \left\{ \vec{x} \in \mathbb{R}^n \middle| \underline{H}\vec{x} \leq \vec{1} \right\},$$

then

$$P^\Delta = \mathrm{conv}(\underline{H}^T).$$



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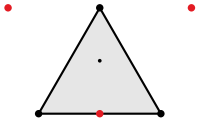

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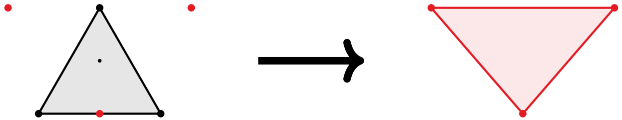
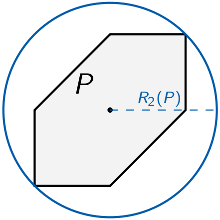
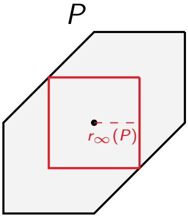


Radii

Circumradius $R_q(P)$: *Minimal* radius of a ball w.r.t. the q-norm, that contains P.

Inradius $r_q(P)$: *Maximal* radius of a ball w.r.t. the q-norm, that is contained in P.





Duality: For the 2-norm: $R_2(P) = \frac{1}{r_2(P^\Delta)}$.

Similarly, for the 1-norm: $R_1(P) = \frac{1}{r_\infty(P^\Delta)}$.

Recall:

$$\|\vec{x}\|_1 = |x_1| + \ldots + |x_n|, \quad \|\vec{x}\|_\infty = \max_i |x_i|.$$

Formulation of the Problem

How to check whether a zonotope $Z_1 = \langle \vec{c}_1, \underline{G}_1 \rangle$ is inside a zonotope $Z_2 = \langle \vec{c}_2, \underline{G}_2 \rangle$?

$$Z_1 \subseteq Z_2$$

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In particular, $r := \frac{1}{d(Z_1,Z_2)}$ is the largest scalar, s.t. $r \cdot Z_1 \subseteq Z_2$.
If Z_1 is the unit hypercube centered around $\vec{c}_1 = \vec{c}_2$, then $r = r_\infty(Z_2)$.

Proving the co-NP-completeness

Theorem (Bodlaender, Gritzmann, Klee, Van Leeuwen)

For parallelotopes Π, computing $R_1(\Pi)$ is NP-hard.

Steps for the reduction:

1. **Input:** Parallelotope Π.

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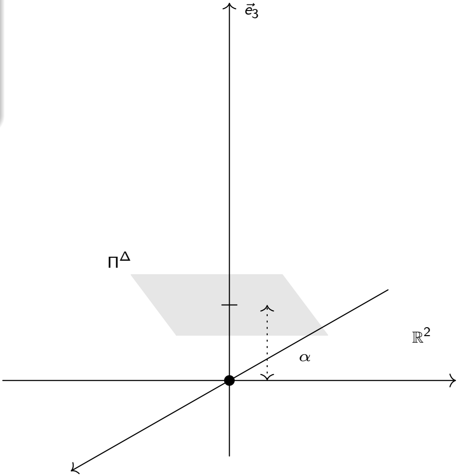

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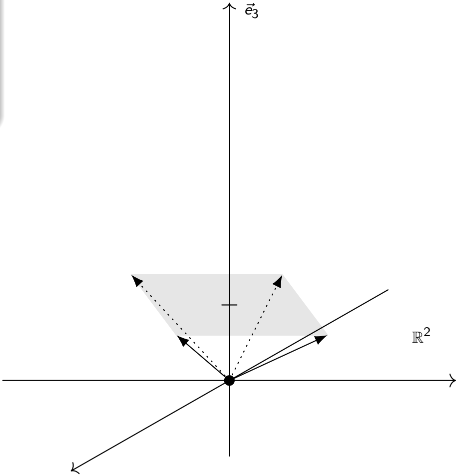

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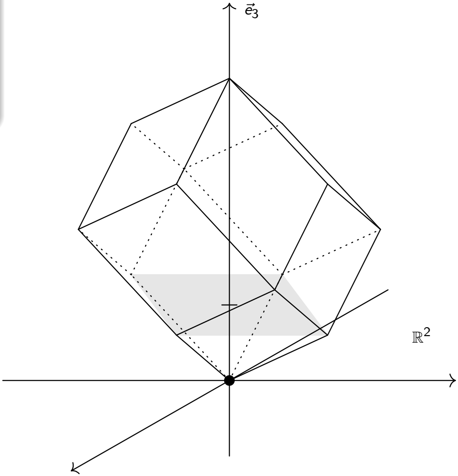
5. Construct a zonotope Z^* that encloses the vertices of Π^Δ.

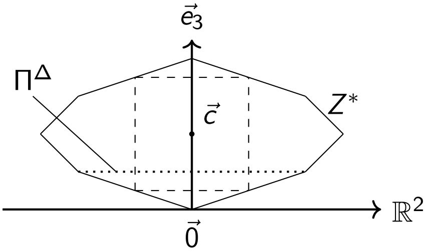


Equivalent Radii

One can prove that

$$r_\infty(Z^*) = c r_\infty(\Pi^\Delta),$$

$\Rightarrow$ Computing $r_\infty(Z^*)$ is as hard as computing $r_\infty(\Pi^\Delta)$, which by duality is as hard as computing $R_1(\Pi)$, which is NP-hard.



Corollary

For any $k > 0$, the problem of checking whether

$$d(Z_1, Z_2) \leq k$$

for zonotopes Z_1, Z_2 is co-NP-complete.

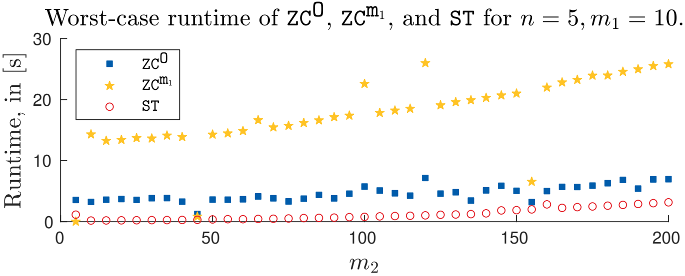
Possible Algorithms for the Zonotope Containment Problem

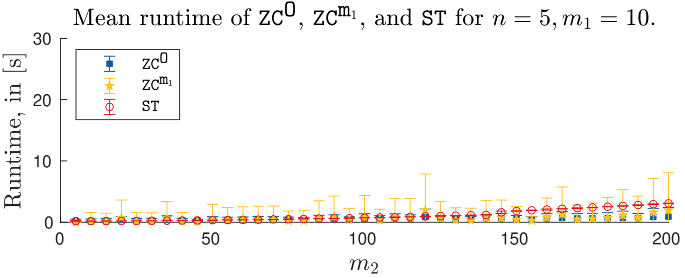
Zonotopes Z_1, Z_2 in $\mathbb{R}^n$, with m_1, m_2 generators, respectively.

- Vertex enumeration of Z_1: Polynomial, if m_1 is fixed $\rightarrow$ $\texttt{ZC}^{\texttt{m}}\texttt{1}$
- Maximal distance to facets: Polynomial, if n or m_2 are fixed $\rightarrow$ $\texttt{ZC}^{\texttt{n}}$, $\texttt{ZC}^{\texttt{m}}\texttt{2}$
- Sadraddini-Tedrake[1]: Polynomial, but not exact $\rightarrow$ $\texttt{ST}$
- Computing $d(Z_1, Z_2)$ using optimization: Here, with fmincon, which can be used to efficiently disprove containment $\rightarrow$ $\texttt{ZC}^{\texttt{0}}$

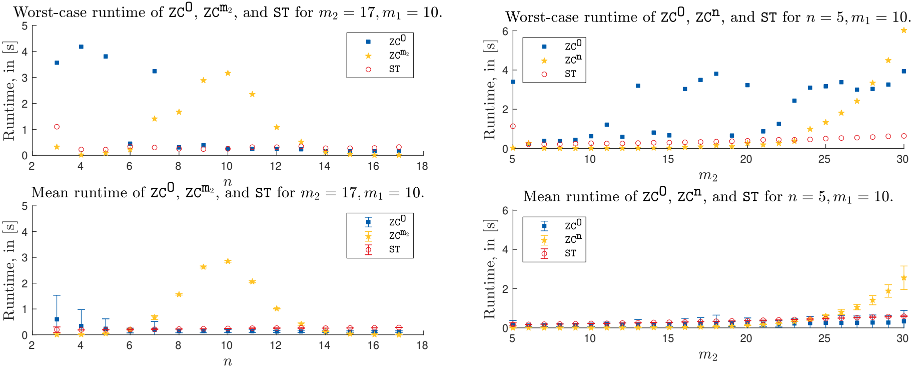
[1] S. Sadraddini and R. Tedrake, "Linear encodings for polytope containment problems," in *IEEE 58th Conference on Decision and Control*, 2019, pp. 4367-4372

Results - Vertex Enumeration

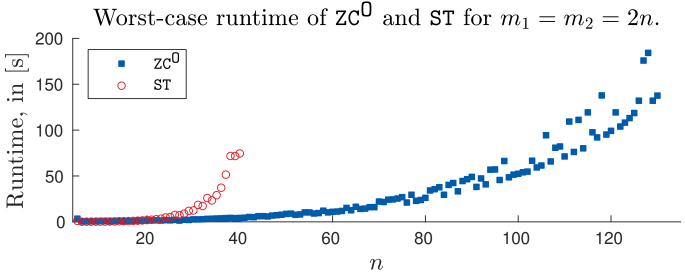


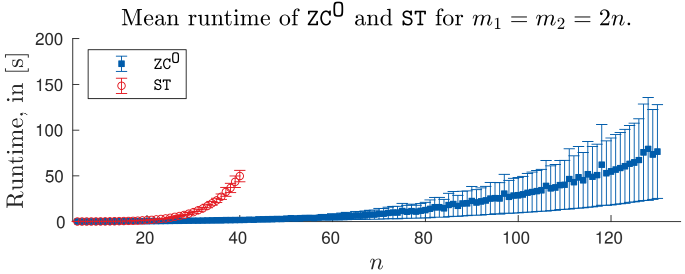


Results - Maximal distance to facets



Results - High Dimensions





Contributions and Open Questions

Contributions:

- Introduction of a new norm induced by any (non-degenerate) zonotope
- Proof of the co-NP-completeness of the zonotope containment problem
- New algorithms that solve the containment problem efficiently in various scenarios

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- Proof of the co-NP-completeness of the zonotope containment problem
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Open questions:

- Is the zonotope containment problem actually co-APX-complete?
- What solver would be best suited for the optimization problem?